GRAPH COLOURINGS, SPACES OF EDGES AND SPACES OF CIRCUITS

CARSTEN SCHULTZ

Abstract. By Lovász' proof of the Kneser conjecture, the chromatic number of a graph G is bounded from below by the index of the $\mathbb{Z}_2$-space $\operatorname{Hom}(K_2, G)$ plus two. We show that the cohomological index of $\operatorname{Hom}(K_2, G)$ is also greater than the cohomological index of the $\mathbb{Z}_2$-space $\operatorname{Hom}(C_{2r+1}, G)$ for $r \geq 1$. This gives a new and simple proof of the strong form of the graph colouring theorem by Babson and Kozlov, which had been conjectured by Lovász, and at the same time shows that it never gives a stronger bound than can be obtained by $\operatorname{Hom}(K_2, G)$. The proof extends ideas introduced by Živaljević in a previous elegant proof of a special case. We then generalise the arguments and obtain conditions under which corresponding results hold for other graphs in place of C_{2r+1}. This enables us to find an infinite family of test graphs of chromatic number 4 among the Kneser graphs.

Our main new result is a description of the $\mathbb{Z}_2$-homotopy type of the direct limit of the system of all the spaces $\operatorname{Hom}(C_{2r+1}, G)$ in terms of the $\mathbb{Z}_2$-homotopy type of $\operatorname{Hom}(K_2, G)$. A corollary is that the coindex of $\operatorname{Hom}(K_2, G)$ does not exceed the coindex of $\operatorname{Hom}(C_{2r+1}, G)$ by more then one if r is chosen sufficiently large. Thus the graph colouring bound in the theorem by Babson & Kozlov is also never weaker than that from Lovász' proof of the Kneser conjecture.

1. Introduction

Background. As a means of proving Kneser's Conjecture, Lovász has shown that a graph is not k-colourable if its neighbourhood complex is $(k-2)$-connected. Since the neighbourhood complex of G is homotopy equivalent to the cell complex $\operatorname{Hom}(K_2, G)$, which was introduced later, this result can be stated as follows. All necessary definitions will be given in the next section.

1.1. **Theorem** (Lovász [Lov78]). *Let G be a graph. Then*

$$\operatorname{conn} \operatorname{Hom}(K_2, G) \leq \chi(G) - 3.$$

The reformulation in terms of the Hom-complex made it natural to ask if similar theorems would hold for graphs other than K_2. In particular, one might have hoped that $\operatorname{conn} \operatorname{Hom}(T, G) \leq \chi(G) - \chi(T) - 1$ for all graphs T and G, provided that $\operatorname{Hom}(T, G) \neq \emptyset$. A graph T such that this holds for all graphs G is called a *test graph* [BK06a]. Hoory and Linial have shown that not every graph is a test graph by giving an example of a graph T with $\operatorname{conn} \operatorname{Hom}(T, K_{\chi(T)}) \geq 0$ [HL05]. Babson and Kozlov succeeded in proving the following positive result that had been conjectured by Lovász.

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1.2. **Theorem** (Babson & Kozlov [BK06b]). *Let G be a graph. Then*

$$\operatorname{conn}\operatorname{Hom}(C_{2r+1}, G) \le \chi(G) - 4.$$

The spaces $\operatorname{Hom}(K_2, G)$ and $\operatorname{Hom}(C_{2r+1}, G)$ are equipped with free $\mathbb{Z}_2$-actions. For such spaces there are several index functions that assign to the space an integer measuring the complexity of the action. We give definitions in 2.2 and recall that

$$\operatorname{conn} X + 1 \le \operatorname{coind}_{\mathbb{Z}_2} X \le \operatorname{cohom-ind}_{\mathbb{Z}_2} X \le \operatorname{ind}_{\mathbb{Z}_2} X$$

for all free $\mathbb{Z}_2$-spaces X. In both of the above theorems, $\operatorname{conn} X$ is only used in the statement for convenience, since it does not depend on the action. In both cases, $\operatorname{conn} X$ can immediately be replaced by $\operatorname{coind}_{\mathbb{Z}_2} X - 1$.

In the case of the original Lovász criterion, the now usual proof yields the following result.

1.3. **Theorem.** *Let G be a graph. Then* $\operatorname{ind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G) \le \chi(G) - 2$.

Proof. A colouring $c\colon G \to K_n$ induces a $\mathbb{Z}_2$-map $\operatorname{Hom}(K_2, G) \to \operatorname{Hom}(K_2, K_n)$, and $\operatorname{Hom}(K_2, K_n)$ is $\mathbb{Z}_2$-homeomorphic to the $(n-2)$-sphere with the antipodal map. For this homeomorphism see e.g. [BK06a, 4.2], [Sch05b, Rem. 2.5], or Example 4.5. $\square$

We remark that the inequality $\operatorname{coind}_{\mathbb{Z}_2} X \le \operatorname{ind}_{\mathbb{Z}_2} X$ that is used to obtain Theorem 1.1 from Theorem 1.3 is essentially the Borsuk-Ulam Theorem.

For C_{2r+1}, Babson and Kozlov had proposed and partially proven the following slightly stronger version of Theorem 1.2.

1.4. **Theorem.** *Let G be a graph. Then* $\operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G) \le \chi(G) - 3$.

Results. The main result of the current work is the following.

1.5. **Theorem.** *Let G be a graph. Then*

$$\operatorname*{colim}_r \operatorname{Hom}(C_{2r+1}, G) \simeq_{\mathbb{Z}_2} \operatorname{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, \operatorname{Hom}(K_2, G)).$$

This result will be part of Theorem 5.11. The left hand side of this homotopy equivalence is the colimit of a diagram that will be defined in Section 5. The right hand side is the space of all equivariant maps from $\mathbb{S}^1$ equipped with the antipodal map to $\operatorname{Hom}(K_2, G)$. This space is made into a $\mathbb{Z}_2$-space via a second $\mathbb{Z}_2$-action on $\mathbb{S}^1$ that is a reflection by a line through the origin in $\mathbb{R}^2$; that $\mathbb{S}^1$ is equipped with these two actions is what the notation $\mathbb{S}^1_b$ indicates (see 4.1).

The space $\operatorname{Hom}(K_2, G)$ can be thought of as the space of oriented edges of G, the $\mathbb{Z}_2$-action being orientation reversal. The space $\operatorname{colim}_r \operatorname{Hom}(C_{2r+1}, G)$ can be thought of as the space of parametrized circuits, or closed paths, in G of arbitrary but odd length, the $\mathbb{Z}_2$-action being the reversal of the direction of the closed paths. The theorem describes that the $\mathbb{Z}_2$-homotopy type of the former determines the $\mathbb{Z}_2$-homotopy type of the latter.

The consequences of this homotopy equivalence for the graph colouring theorems above is described by the following corollaries. Since in the body of this article they will be proved before the theorem, we here give their derivations from it. Readers who are not very familiar with the methods are encouraged to skip them.

1.6. **Corollary.** *Let G be a graph with at least one edge. Then*

$$\operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G) + 1 \le \operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G).$$

Proof. We have a composition of $\mathbb{Z}_2$-maps

$$\begin{aligned}\mathbb{S}^1_b \times_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G) &\to_{\mathbb{Z}_2} \mathbb{S}^1_b \times_{\mathbb{Z}_2} \operatorname*{colim}_r \operatorname{Hom}(C_{2r+1}, G)\\ &\simeq_{\mathbb{Z}_2} \mathbb{S}^1_b \times_{\mathbb{Z}_2} \operatorname{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, \operatorname{Hom}(K_2, G))\\ &\to_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G),\end{aligned}$$

where the last arrow is induced by evaluation. The result now follows from Lemma 4.2. □

1.7. **Corollary.** *Let G be a graph with at least one edge. Then*

$$\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G) \leq \lim_{r\to\infty} \operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G) + 1.$$

Proof. Assume $\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G) \geq k+1$, i.e. the existence of a $\mathbb{Z}_2$-map $\mathbb{S}^{k+1} \to_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G)$. There is a map $\mathbb{S}^1_b \times_{\mathbb{Z}_2} \mathbb{S}^k \to_{\mathbb{Z}_2} \mathbb{S}^{k+1}$ and hence a map $\mathbb{S}^1_b \times_{\mathbb{Z}_2} \mathbb{S}^k \to_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G)$. This means that there is a map

$$\mathbb{S}^k \to_{\mathbb{Z}_2} \operatorname{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, \operatorname{Hom}(K_2, G)) \simeq_{\mathbb{Z}_2} \operatorname*{colim}_r \operatorname{Hom}(C_{2r+1}, G).$$

Because of the compactness of $\mathbb{S}^k$, it follows that whenever r is large enough there is a map $\mathbb{S}^k \to_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G)$, i.e. $\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G) \geq k$. □

Outline. We start with necessary definitions and facts in Section 2.

In Section 3 we give a short proof of Corollary 1.6. This reduces Theorem 1.4 to Theorem 1.3. So far, the proof of Theorem 1.4 in [Sch05a] had also been the simplest known proof of Theorem 1.2. However, Živaljević had given a surprising proof of

$$\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G) \leq 2\left\lceil \frac{\chi(G)}{2} \right\rceil - 3$$

that was even simpler [Živ05b, Živ05a]. Our proof in Section 3 extends arguments from Živaljević's proof. If one is content to obtain a bound on the coindex of $\operatorname{Hom}(C_{2r+1}, G)$ instead of the cohomological index, it is completely elementary in the sense that the Algebraic Topology used is not more advanced than the degree of maps between spheres of the same dimension, i.e. nothing more advanced than the Borsuk-Ulam Theorem.

In Section 4 we generalize the proof and obtain conditions under which inequalities similar to Corollary 1.6 hold. This culminates in Theorem 4.9 which contains Corollary 1.6 as a special case. As an application we show in Example 4.13 that there is an infinite family of Kneser graphs with chromatic number 4 which are test graphs. Except for a topological lemma at its beginning, this section is not needed for what follows.

In Section 5 we first prove Corollary 1.7 and finally Theorem 1.5.

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2. Objects of study

We introduce the objects and concepts used in the proof. The only thing worth to be mentioned specifically is Definition 2.9, which is very natural but to our knowledge has not been used explicitly before.

Free $\mathbb{Z}_2$-spaces. We assume all spaces to be CW-spaces. A good introduction to equivariant methods from the point of view of combinatorial applications is [Mat03].

2.1. **Definition.** For an integer $m \geq -1$, we say that a topological space X is m-connected if every continuous map $\mathbb{S}^k \to X$ with $-1 \leq k \leq m$ can be extended to a continuous map $\mathbb{D}^{k+1} \to X$. We define the *connectivity* of X, $\operatorname{conn} X \in \mathbb{Z} \cup \{\infty\}$, to be the largest m such that X is m-connected.

2.2. **Definition.** Let X be a free $\mathbb{Z}_2$-space, i.e. a space with a fixed point free involution. We define the index and coindex of X by

$$\operatorname{ind}_{\mathbb{Z}_2} X := \min\left\{k\colon \text{There is a } \mathbb{Z}_2\text{-map } X \to \mathbb{S}^k\right\},$$
$$\operatorname{coind}_{\mathbb{Z}_2} X := \max\left\{k\colon \text{There is a } \mathbb{Z}_2\text{-map } \mathbb{S}^k \to X\right\}.$$

Since there is a $\mathbb{Z}_2$-map $f\colon X \to \mathbb{S}^\infty$ and this map is unique up to $\mathbb{Z}_2$-homotopy, we can also define the cohomological index

$$\operatorname{cohom-ind}_{\mathbb{Z}_2} X := \max\left\{k\colon \bar{f}^*(\gamma^k) \neq 0\right\},$$

where $\bar{f}\colon X/\mathbb{Z}_2 \to \mathbb{R}\mathrm{P}^\infty$ is the map induced by f, and $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[\gamma]$.

2.3. **Definition.** If X, Y are $\mathbb{Z}_2$-spaces and $f\colon X \to Y$ a map, then we will also call f an *odd map* if it is equivariant, and we will call f an *even map* if it maps each orbit to a single point.

2.4. **Proposition.** *If X is a free $\mathbb{Z}_2$-space and $p_X\colon X \to X/\mathbb{Z}_2$ the canonical quotient map, then there is the* cohomology transfer maps $p_X^!\colon H^*(X;\mathbb{Z}_2) \to H^*(X/\mathbb{Z}_2;\mathbb{Z}_2)$ *which fits in a long exact sequence*

$$H^k(X/\mathbb{Z}_2;\mathbb{Z}_2) \xrightarrow{p_X^*} H^k(X;\mathbb{Z}_2) \xrightarrow{p_X^!} H^k(X/\mathbb{Z}_2;\mathbb{Z}_2)$$
$$\xrightarrow{\delta^*} H^{k+1}(X/\mathbb{Z}_2;\mathbb{Z}_2) \xrightarrow{p_X^*} H^{k+1}(X;\mathbb{Z}_2)$$

which is natural with respect to $\mathbb{Z}_2$-maps. For $X = \mathbb{S}^\infty$ and $k \geq 0$ it follows that $\delta^*\colon H^k(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \to H^{k+1}(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$ is an isomorphism, i.e. $\delta^*(\gamma^k) = \gamma^{k+1}$.

Introductions to transfer maps which are sufficient for our purposes when translated from homology to cohomology can be found in the proof of the Borsuk-Ulam Theorem presented in the textbooks of Bredon [Bre93, pp. 240–241] and Hatcher [Hat02, p. 174]. A more complete reference is [Bre72, Chap. III].

Order complexes. For a partially ordered set, or *poset*, P, we denote by ΔP its *order complex*, the simplicial complex with vertex set P that consists of all chains in P. Any monotone (or antitone) map $f\colon P \to Q$ between posets induces a simplicial map $\Delta f\colon \Delta P \to \Delta Q$. For a cell complex C we denote its *face poset* by FC and its underlying space by $|C|$. Thus $\Delta(FC)$ is the barycentric subdivision of the complex C and $|\Delta(FC)| \approx |C|$. For posets P and Q, $\Delta(P \times Q)$ is a simplicial subdivision of the cell complex $\Delta P \times \Delta Q$, one often used for the product of simplicial complexes with vertex orderings.

For posets P and Q we denote by $\operatorname{Mon}(P,Q)$ the poset of all order preserving maps from P to Q.

Graph complexes. We will be brief in our description of Hom-complexes. A good introduction is contained in [Koz06].

All graphs that we consider are finite, simple, and without loops. The vertex set of a graph G is denoted by $V(G)$, the set of edges by $E(G)$.

2.5. **Notation.** Let $n \in \mathbb{N}$. K_n denotes the complete graph on n vertices with vertex set $\{0, \ldots, n-1\}$. C_n is the cycle of length n with vertex set $\{0, \ldots, n-1\}$.

2.6. **Definition.** A *graph homomorphism* from G to H is a function $f\colon V(G) \to V(H)$ that respects the edge relation, i.e. such that $\{f(u), f(u')\} \in E(H)$ whenever $\{u, u'\} \in E(G)$. The set of all graph homomorphisms from G to H is denoted by $\mathrm{Hom}_0(G, H)$.

2.7. **Definition.** Let G, H be graphs. A *multi-homomorphism* from G to H is a function $\phi\colon V(G) \to \mathcal{P}(V(H)) \setminus \{\emptyset\}$ such that every function $f\colon V(G) \to V(H)$ with $f(v) \in \phi(v)$ for all $v \in V(G)$ is a graph homomorphism.

2.8. **Definition.** Let G, H be graphs. A function $\phi\colon V(G) \to \mathcal{P}(V(H)) \setminus \{\emptyset\}$ can be identified with a cell of the cell complex

$$\prod_{v \in V(G)} \Delta^{\#V(H)-1}.$$

The subcomplex of all cells indexed by multi-homomorphisms is denoted by $\mathrm{Hom}(G, H)$. We identify elements of $F\,\mathrm{Hom}(G, H)$ with the corresponding multi-homomorphisms and $\mathrm{Hom}_0(G, H)$ with the 0-skeleton of $\mathrm{Hom}(G, H)$.

2.9. **Definition and Proposition.** *The monotone map*

$$\begin{aligned} *\colon F\,\mathrm{Hom}(G, G') \times F\,\mathrm{Hom}(G', G'') \;&\longrightarrow\; F\,\mathrm{Hom}(G, G'') \\ (\phi * \rho)(v) \;&:=\; \rho[\phi(v)] \end{aligned}$$

induces a continuous map

$$*\colon |\mathrm{Hom}(G, G')| \times |\mathrm{Hom}(G', G'')| \;\longrightarrow\; |\mathrm{Hom}(G, G'')|.$$

This map is associative and its restriction to $\mathrm{Hom}_0(G, G') \times \mathrm{Hom}_0(G', G'') \to \mathrm{Hom}_0(G, G'')$ *coincides with composition of graph homomorphisms. Hence its restrictions* $\mathrm{Hom}_0(G, G') \times \mathrm{Hom}(G', G'') \to \mathrm{Hom}(G, G')$ *and* $\mathrm{Hom}(G, G') \times \mathrm{Hom}_0(G', G'') \to \mathrm{Hom}(G, G')$ *make* Hom *into a functor, contravariant in the first and covariant in the second argument.*

2.10. **Definition and Proposition.** *If H is a graph and $\alpha \in \mathrm{Hom}_0(H, H)$ satisfies $\alpha^2 = \mathrm{id}_G$ and α flips an edge of H (i.e. the edge is invariant but not fixed under α), then for every graph G*

$$\begin{gathered} |\mathrm{Hom}(H, G)| \to |\mathrm{Hom}(H, G)| \\ x \mapsto \alpha * x \end{gathered}$$

is a fixed point free involution. It is in this way that we make $\mathrm{Hom}(K_2, G)$ *and* $\mathrm{Hom}(C_n, G)$ *into free $\mathbb{Z}_2$-spaces.*

3. Proof of Corollary 1.6

Let G be a graph, $r \geq 1$. We consider the free $\mathbb{Z}_2$-actions on $\operatorname{Hom}(K_2, G)$ and $\operatorname{Hom}(C_{2r+1}, G)$ induced by $\alpha \in \operatorname{Hom}_0(K_2, K_2)$ and $\beta \in \operatorname{Hom}_0(C_{2r+1}, C_{2r+1})$ with $\alpha(v) = 1 - v$ and $\beta(v) = 2r - v$.

Every $x \in |\operatorname{Hom}(K_2, C_{2r+1})|$ induces a map

$$\begin{aligned} f_x\colon |\operatorname{Hom}(C_{2r+1}, G)| &\to |\operatorname{Hom}(K_2, G)|, \\ y &\mapsto x * y. \end{aligned}$$

We define $x_0, x_1 \in F\operatorname{Hom}(K_2, C_{2r+1})$ by $x_0(0) := \{r\}$, $x_0(1) := \{r-1, r+1\}$, $x_1(0) = \{0\}$, $x_1(1) = \{2r\}$. Then $x_0 * \beta = x_0$ and $x_1 * \beta = \alpha * x_1$. Hence, $f_{x_0}(\beta * y) = f_{x_0}(y)$ and $f_{x_1}(\beta * y) = \alpha * f_{x_1}(y)$ for all y, i.e., using the language of Definition 2.3, f_{x_0} is even and f_{x_1} odd.

It is easy to check that the complex $\operatorname{Hom}(K_2, C_{2r+1})$ is homeomorphic to a 1-sphere and in particular path-connected. A path (x_t) from x_0 to x_1 gives a homotopy f_{x_t}, and hence $f_{x_0} \simeq f_{x_1}$.

The inequality $\operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G) + 1 \leq \operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G)$ now is a consequence of the following Lemma. $\square$

3.1. Lemma. *Let X, Y be free $\mathbb{Z}_2$-spaces, $Y \neq \emptyset$, $f, g\colon X \to Y$ maps. If f is odd, g even, and $f \simeq g$, then* $\operatorname{cohom-ind}_{\mathbb{Z}_2} X + 1 \leq \operatorname{cohom-ind}_{\mathbb{Z}_2} Y$.

Before proving the Lemma we quickly show how to obtain the weaker inequality $\operatorname{coind}_{\mathbb{Z}_2} X + 1 \leq \operatorname{ind}_{\mathbb{Z}_2} Y$. If for some k there is a $\mathbb{Z}_2$-map $k\colon Y \to \mathbb{S}^k$, then the existence of a $\mathbb{Z}_2$-map $l\colon \mathbb{S}^k \to X$ would give rise to an odd map $k \circ f \circ l\colon \mathbb{S}^k \to \mathbb{S}^k$ and an even map $k \circ g \circ l$. These maps would be homotopic, which contradicts that even maps between spheres have even degree and odd maps have odd degree [Hat02, Prop. 2B.6].

Proof. Let $k \geq 0$. We show that $\operatorname{cohom-ind}_{\mathbb{Z}_2} Y \leq k$ implies $\operatorname{cohom-ind}_{\mathbb{Z}_2} X < k$. Assume that $\bar{h}^*(\gamma^{k+1}) = 0$ where $\bar{h}$ is induced by a $\mathbb{Z}_2$-map $h\colon Y \to \mathbb{S}^\infty$. From Proposition 2.4 we obtain a commutative diagram

$$\begin{tikzcd}
 & & H^k(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \arrow[r, "\delta^*", "\cong"'] \arrow[d, "\bar{h}^*"] & H^{k+1}(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \arrow[d, "\bar{h}^*"] \\
 & H^k(Y;\mathbb{Z}_2) \arrow[r, "p_Y^!"] \arrow[d, "f^*"] \arrow[dl, "(g')^*"'] & H^k(Y/\mathbb{Z}_2;\mathbb{Z}_2) \arrow[r, "\delta^*"] \arrow[d, "\bar{f}^*"] & H^{k+1}(Y/\mathbb{Z}_2;\mathbb{Z}_2) \\
H^k(X/\mathbb{Z}_2;\mathbb{Z}_2) \arrow[r, "p_X^*"] & H^k(X;\mathbb{Z}_2) \arrow[r, "p_X^!"] & H^k(X/\mathbb{Z}_2;\mathbb{Z}_2) &
\end{tikzcd}$$

with exact rows, where g' is defined by $g = g' \circ p_X$. Now

$$\delta^*(\bar{h}^*(\gamma^k)) = \bar{h}^*(\delta^*(\gamma^k)) = \bar{h}^*(\gamma^{k+1}) = 0.$$

Therefore there is an $\eta \in H^k(Y;\mathbb{Z}_2)$ with $\bar{h}^*(\gamma^k) = p_Y^!(\eta)$ and

$$(\bar{h} \circ \bar{f})^*(\gamma^k) = \bar{f}^*(\bar{h}^*(\gamma^k)) = \bar{f}^*(p_Y^!(\eta)) = p_X^!(f^*(\eta)) = p_X^!(p_X^*((g')^*(\eta))) = 0$$

follows. $\square$

4. Generalizations

In Section 3 we have shown that

$$\text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, H) + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, H),$$

and the proof used properties of $\operatorname{Hom}(K_2, C_{2r+1})$. We will generalize this and determine properties of $\operatorname{Hom}(G, G')$ which imply that

$$\text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(G', H) + k \leq \text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(G, H)$$

for a suitable $k \geq 1$ and all graphs H, Theorem 4.9. As an application we obtain an infinite family of test graphs with chromatic number 4 in Example 4.13.

For the complex $\operatorname{Hom}(K_2, C_{2r+1})$ we implicitly used the $\mathbb{Z}_2$-operation induced by the involution of C_{2r+1} as well as that induced by the involution of K_2. It is easily seen that there is a homeomorphism $\operatorname{Hom}(K_2, C_{2r+1}) \approx \mathbb{S}^1$ under which the operation induced by the involution of C_{2r+1} is equivalent to a reflection by a line through the origin of $\mathbb{R}^2 \supset \mathbb{S}^1$ and the operation induced by the involution of K_2 is equivalent to the antipodal action on $\mathbb{S}^1$. This leads us to the following definition.

4.1. **Notation.** We write $\mathbb{Z}_2$ multiplicatively as $\mathbb{Z}_2 = \{1, \tau\}$. We will always consider $\mathbb{S}^k$ to be equipped with the left $\mathbb{Z}_2$ action given by the antipodal map

$$\tau \cdot (x_0, \ldots, x_k) := (-x_0, \ldots, -x_k).$$

When additionally equipped with the right $\mathbb{Z}_2$-action by the reflection

$$(x_0, \ldots, x_k) \cdot \tau := (-x_0, x_1, \ldots, x_k)$$

we will write the sphere as $\mathbb{S}^k_b$. Since these two actions commute, we can also see them as a single $\mathbb{Z}_2 \times \mathbb{Z}_2$-action.

The case $k = 1$ of the following lemma is equivalent to Lemma 3.1.

4.2. **Lemma.** *Let $X \neq \emptyset$ be a free $\mathbb{Z}_2$-space and $k \geq 0$. Then*

$$\mathbb{S}^k_b \times_{\mathbb{Z}_2} X := (\mathbb{S}^k_b \times X)/_{(s,\tau x)\sim(s\tau,x)}$$

is also a free $\mathbb{Z}_2$-space, and

$$\text{cohom-ind}_{\mathbb{Z}_2} X + k \leq \text{cohom-ind}_{\mathbb{Z}_2}(S^k \times_{\mathbb{Z}_2} X).$$

Proof. Since the left and right action on $\mathbb{S}^k_b$ commute, the left action induces an action on $\mathbb{S}^k_b \times_{\mathbb{Z}_2} X$, which is free, because the left actions on $\mathbb{S}^k_b$ and X are free. Furthermore $\mathbb{S}^0_b \times_{\mathbb{Z}_2} X \approx_{\mathbb{Z}_2} X$, and it will be sufficient to show

$$\text{cohom-ind}_{\mathbb{Z}_2}(\mathbb{S}^{k-1}_b \times_{\mathbb{Z}_2} X) + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2}(\mathbb{S}^k_b \times_{\mathbb{Z}_2} X)$$

for $k \geq 1$.

We consider the map

$$\begin{aligned} f \colon \mathbb{S}^{k-1} \times [0,1] &\to \mathbb{S}^k \\ (s,t) &\mapsto (\cos(t\pi/2)s, \sin(t\pi/2)) . \end{aligned}$$

This map satisfies $f(s \cdot \tau, t) = f(s,t) \cdot \tau$, $f(\tau \cdot s, 0) = \tau \cdot f(s, 0)$, $f(\tau \cdot s, 1) = f(s, 1)$ for all $s \in \mathbb{S}^{k-1}$ and $t \in [0,1]$. It therefore induces a homotopy

$$\begin{aligned} (\mathbb{S}^{k-1}_b \times_{\mathbb{Z}_2} X) \times [0,1] &\to \mathbb{S}^k_b \times_{\mathbb{Z}_2} X \\ ([(s,x)], t) &\mapsto [(f(s,t), x)] \end{aligned}$$

from an odd to an even map and Lemma 3.1 can be applied. $\square$

4.3. **Notation.** If G and G' are graphs, then we write the $\mathbb{Z}_2$-action on $\operatorname{Hom}(G, G')$ induced by an involution of G as multiplication from the left with elements of $\mathbb{Z}_2$, and the $\mathbb{Z}_2$-action induced by an involution of G' as right multiplication. Again, these two actions commute because of the associativity of $*$.

4.4. **Theorem.** *Let G, G' be graphs with edge-flipping involutions. If there exists a $\mathbb{Z}_2 \times \mathbb{Z}_2$-map $f\colon \mathbb{S}^k_b \to \operatorname{Hom}(G, G')$, then*

$$\operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(G', H) + k \leq \operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(G, H)$$

for all graphs H with $\operatorname{Hom}(G', H) \neq \emptyset$.

Proof. The diagram

$$\begin{tikzcd}
\mathbb{S}^k_b \times \operatorname{Hom}(G', H) \arrow[r, "f\times\mathrm{id}"] \arrow[d] & \operatorname{Hom}(G, G') \times \operatorname{Hom}(G', H) \arrow[r, "*"] \arrow[d] & \operatorname{Hom}(G, H) \\
\mathbb{S}^k_b \times_{\mathbb{Z}_2} \operatorname{Hom}(G', H) \arrow[r, "\mathbb{Z}_2"'] & \operatorname{Hom}(G, G') \times_{\mathbb{Z}_2} \operatorname{Hom}(G', H) \arrow[ur, "\mathbb{Z}_2"'] &
\end{tikzcd}$$

commutes. The existence of the $\mathbb{Z}_2$-map making the right triangle commutative follows from the associativity of $*$. The two $\mathbb{Z}_2$-maps at the bottom of the diagram show that

$$\operatorname{cohom-ind}_{\mathbb{Z}_2}(\mathbb{S}^k \times_{\mathbb{Z}_2} \operatorname{Hom}(G', H)) \leq \operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(G, H),$$

and Lemma 4.2 concludes the proof. $\square$

4.5. **Example.** As mentioned above, we indeed have $\operatorname{Hom}(K_2, C_{2r+1}) \approx_{\mathbb{Z}_2\times\mathbb{Z}_2} \mathbb{S}^1_b$, so that Corollary 1.6 is a special case of Theorem 4.4. Similarly, if we equip K_n, $n > 2$, with the $\mathbb{Z}_2$-action flipping $\{0, 1\}$ and keeping the other vertices fixed, then $\operatorname{Hom}(K_2, K_n) \approx_{\mathbb{Z}_2\times\mathbb{Z}_2} \mathbb{S}^{n-2}_b$ for $n \geq 0$. This can be seen as follows. If for a non-empty subset A of $V(K_n)$ we let b_A denote the barycentre of the corresponding face of the $(n-1)$-simplex, then mapping $\phi \in F \operatorname{Hom}(K_2, K_n)$ to $\frac{1}{2} b_{\phi(0)} + \frac{1}{2} b_{\mathcal{C}\phi(1)}$ we obtain a homeomorphism from $\operatorname{Hom}(K_2, K_n)$ to the boundary of the (n-1)-simplex. This sends the left action on $\operatorname{Hom}(K_2, K_n)$ to the antipodal map. The right action on $\operatorname{Hom}(K_2, K_n)$ is sent to the map on the simplex induced by exchanging the vertices 0 and 1. This is a reflection by the affine subspace spanned by $\left\{b_{\{0,1\}}, b_2, \ldots, b_{n-1}\right\}$.

We will take up these examples again in 4.11 and 4.12.

We would like to have a version of Theorem 4.4 with conditions that are easier to check. A $\mathbb{Z}_2 \times \mathbb{Z}_2$-map $\mathbb{S}^k_b \to \operatorname{Hom}(G, G')$ maps the fixed point sets in $\mathbb{S}^k_b$ of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the corresponding fixed point sets in $\operatorname{Hom}(G, G')$. Since the left action of $\mathbb{Z}_2$ on $\mathbb{S}^k_b$ is free, the fixed point set of $(\tau, 1)$ in $\mathbb{S}^k_b$ is empty. The fixed point set in $\operatorname{Hom}(G, G')$ of (τ, τ) corresponds to the equivariant multi-homomorphisms from G to G'.

4.6. **Definition.** For $i \in \{0, 1\}$ let G_i be a graph equipped with a $\mathbb{Z}_2$-action given by $\alpha_i \in \operatorname{Hom}_0(G_i, G_i)$ with $\alpha_i^2 = \mathrm{id}$. We define $\operatorname{Hom}_{\mathbb{Z}_2}(G_0, G_1)$ to be the subspace of $|\operatorname{Hom}(G_0, G_1)|$ consisting of all x with $\alpha_0 * x = x * \alpha_1$.

The fixed-point set in $\operatorname{Hom}(G, G')$ of $(1, \tau)$ surely contains all the multi-homomorphisms whose image is contained in the induced subgraph of G on all vertices which are fixed by the involution. However, since we are dealing with multi-homomorphism and

not only homomorphisms, the fixed point set can be larger. This leads us to the following definition.

4.7. **Definition.** Let G be a graph equipped with a $\mathbb{Z}_2$-action given by a homomorphism $\alpha\colon G \to G$ with $\alpha^2 = \mathrm{id}$. We define a graph $G^{\mathbb{Z}_2}$ by

$$V(G^{\mathbb{Z}_2}) := \{\{u, \alpha(u)\} : u \in V(G)\}$$
$$E(G^{\mathbb{Z}_2}) := \{\{\{u, \alpha(u)\}, \{v, \alpha(v)\}\} : \{u, v\} \in E(G) \text{ and } \{u, \alpha(v)\} \in E(G)\}$$

We also define $\iota_G \in F\operatorname{Hom}(G^{\mathbb{Z}_2}, G)$ by $\iota_G(v) := v$.

The graph $G^{\mathbb{Z}_2}$ is determined by the following universal property.

4.8. **Proposition.** *Let G be graph and α an involution on G. $\iota_G\colon G^{\mathbb{Z}_2} \to G$ is a multi-homomorphism with $\iota_G * \alpha = \iota_G$. If H is a graph and $h\colon H \to G$ is a multi-homomorphism with $h * \alpha = h$, then there is a unique multi-homomorphism $g\colon H \to G^{\mathbb{Z}_2}$ with $h = g * \iota_G$.* □

The analogous property with multi-homomorphisms replaced by homomorphisms is fulfilled by the inclusion of the induced subgraph of G on all vertices which are fixed by the involution.

We can now state a more practical, but slightly weaker, form of Theorem 4.4.

4.9. **Theorem.** *Let G, G' be graphs with $\mathbb{Z}_2$-actions, the action on G flipping an edge, and $k \geq 1$. If*

- ▷ $\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(G, G'^{\mathbb{Z}_2}) \geq k - 1$,
- ▷ $\operatorname{Hom}_{\mathbb{Z}_2}(G, G') \neq \emptyset$, *and*
- ▷ $\operatorname{Hom}(G, G')$ *is* $(k-1)$*-connected,*

then

$$\operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(G', H) + k \leq \operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(G, H)$$

for all graphs H with $\operatorname{Hom}(G', H) \neq \emptyset$.

Proof of Theorem 4.9. Since $\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(G, G'^{\mathbb{Z}_2}) \geq k - 1$, there is a map

$$h\colon \mathbb{S}^{k-1} \to_{\mathbb{Z}_2} \operatorname{Hom}(G, G'^{\mathbb{Z}_2}) \xrightarrow{\operatorname{Hom}(G, \iota_{G'})} \operatorname{Hom}(G, G').$$

This map satisfies $h(-s) = \tau h(s)$ and $h(s)\tau = h(s)$ for all $s \in \mathbb{S}^{k-1}$. We also choose $y \in \operatorname{Hom}_{\mathbb{Z}_2}(G, G') \subset |\operatorname{Hom}(G, G')|$, that is $\tau y \tau = y$. Since $\operatorname{Hom}(G, G')$ is $(k-1)$-connected, and the only fixed point of the action $s \mapsto -s$ on $\mathbb{D}^k$ is the origin, these choices can be extended to a map $g\colon \mathbb{D}^k \to \operatorname{Hom}(G, G')$ with $g(-s) = \tau g(s)\tau$ for all $s \in \mathbb{D}^k$, $g(0) = y$, and $g|_{\mathbb{S}^{k-1}} = h$. We now define a map

$$f\colon \mathbb{S}^k_b \to \operatorname{Hom}(G, G')$$
$$(x_0, \ldots, x_k) \mapsto \begin{cases} g(x_1, \ldots, x_k), & x_0 \geq 0, \\ g(x_1, \ldots, x_k) \cdot \tau, & x_0 \leq 0. \end{cases}$$

This map commutes with the left and right actions, so that Theorem 4.4 is applicable. □

4.10. **Corollary.** *Let T be a graph with a $\mathbb{Z}_2$-action that flips an edge and $k \geq 1$. If* $\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, T^{\mathbb{Z}_2}) \geq k - 1$ *and* $\operatorname{Hom}(K_2, T)$ *is $(k-1)$-connected, then*

$$\operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(T, H) + k \leq \operatorname{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, H)$$

for all graphs H with $\mathrm{Hom}(T, H) \neq \emptyset$*. It follows that*

$$\chi(G) \geq \text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(T, G) + k + 2$$

for all graphs G with $\mathrm{Hom}(T, G) \neq \emptyset$*. In particular, if $k = \chi(T) - 2$ then T is a test graph as defined in the introduction.*

Proof. For the first equation we set $G = K_2$, $G' = T$ in the Theorem. The edge of T that is flipped by the action ensures that $\mathrm{Hom}_{\mathbb{Z}_2}(K_2, T) \neq \emptyset$. Now for a graph G with $\mathrm{Hom}(T, G) \neq \emptyset$, we set $H = K_{\chi(G)}$ to obtain

$$\begin{aligned} \text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(T, G) + k \leq \text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(T, K_{\chi(G)}) + k \leq \\ \leq \text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(K_2, K_{\chi(G)}) = \chi(G) - 2 \end{aligned}$$

and hence the second equation. □

4.11. **Example** (Odd circuits). In Section 3 we have dealt with C_{2r+1}, $r \geq 1$. The result could also have been achieved by applying Corollary 4.10 with $k = 1 = \chi(C_{2r+1}) - 2$. The multi-homomorphism $x_0 \in F\,\mathrm{Hom}(K_2, C_{2r+1})$ used there shows that $\mathrm{Hom}(K_2, C_{2r+1}^{\mathbb{Z}_2}) \neq \emptyset$. Indeed, $C_{2r+1}^{\mathbb{Z}_2}$ has the single edge $\{\{r-1, r+1\}, \{r\}\}$.

4.12. **Example** (Complete graphs). If we equip K_n, $n > 2$ with the $\mathbb{Z}_2$-action flipping $\{0, 1\}$ and keeping the other vertices fixed, then we can apply Corollary 4.10 with $k = n-2$, since $K_n^{\mathbb{Z}_2} \cong K_{n-1}$. That K_n is a test graph has been shown in [BK06a]. In contrast to the case of C_{2r+1}, it will probably not come as a surprise that other complete graphs do not yield better bounds on chromatic numbers than K_2.

Instead of applying Corollary 4.10 with $k = n - 2$ to obtain

$$\text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(K_n, H) + n - 2 \leq \text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(K_2, H)$$

we can also apply Theorem 4.9 with $k = 1$ to abtain the stronger result

$$\text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(K_n, H) + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(K_{n-1}, H).$$

The necessary fact that $\mathrm{Hom}(K_{n-1}, K_n)$ is path-connected follows by observing that the group of permutations of $V(K_n)$ is generated by transpositions of the form $(i, n-1)$. More generally, $\mathrm{Hom}(K_m, K_n)$ is homotopy equivalent to a wedge of $(n-m)$-spheres for $n \geq m$ [BK06a].

4.13. **Example** (Kneser graphs). Let $KG_{n,l}$ denote the Kneser graph of n-element subsets of $\{0, \ldots, 2n + l - 1\}$. Edges are pairs of disjoint sets. It is the result of [Lov78] that $\mathrm{Hom}(K_2, KG_{n,l})$ is $(l-1)$-connected and $\chi(KG_{n,l}) = l + 2$.

Let $r, s \geq 1$. On the set $\{0, \ldots, 4r + 2s - 1\}$ we consider the permutation

$$\sigma\colon i \mapsto 4r + 2s - 1 - i.$$

This induces a $\mathbb{Z}_2$-action on $KG_{2r,2s}$ which flips the edge

$$\{\{0, \ldots, 2r - 1\}, \{2r + 2s, \ldots, 4r + 2s - 1\}\}.$$

There is a homomorphism $KG_{r,s} \to KG_{2r,2s}^{\mathbb{Z}_2}$ given by $M \mapsto \{M \cup \sigma[M]\}$. Therefore $\mathrm{coind}_{\mathbb{Z}_2} \mathrm{Hom}(K_2, KG_{2r,2s}^{\mathbb{Z}_2}) \geq \mathrm{coind}_{\mathbb{Z}_2} \mathrm{Hom}(K_2, KG_{r,s}) = s$, and Corollary 4.10 can be applied with $k = s + 1$.

For $s = 1$, this yields that $KG_{2r,2}$ is a test graph with chromatic number 4. For $r > 1$, it is triangle-free. So far, the only known test graph with chromatic number 4 was K_4. There are also good candidates in [Živ05a]; these are built from triangles.

5. The colimits of $\operatorname{Hom}(C_{2r+1}, G)$ and $\operatorname{Hom}(C_{2r}, G)$ for $r \to \infty$

We fix a graph G.

5.1. **Definition.** Let $m \geq 3$. We define a monotone map

$$\begin{aligned}\eta_m\colon \operatorname{F\,Hom}(C_m, G) &\to_{\mathbb{Z}_2} \operatorname{Mon}_{\mathbb{Z}_2}(\operatorname{F\,Hom}(K_2, C_m), \operatorname{F\,Hom}(K_2, G)),\\ \eta_m(\phi)(\rho) &:= \rho * \phi.\end{aligned}$$

This map, or rather its adjoint

$$\operatorname{F\,Hom}(K_2, C_m) \times_{\mathbb{Z}_2} \operatorname{F\,Hom}(C_m, G) \to_{\mathbb{Z}_2} \operatorname{F\,Hom}(K_2, G),$$

has been vital in Section 3. We will now define a map that will induce a homotopy inverse in the limit.

5.2. **Definition.** Let $m \geq 3$. We define a monotone map

$$\begin{aligned}\theta_m\colon \operatorname{Mon}_{\mathbb{Z}_2}(\operatorname{F\,Hom}(K_2, C_m), \operatorname{F\,Hom}(K_2, G)) &\to_{\mathbb{Z}_2} \operatorname{F\,Hom}(C_{3m}, G),\\ \theta_m(f)(3k) &:= f((\{k\}, \{k-1\}))_2,\\ \theta_m(f)(3k+1) &:= f((\{k\}, \{k-1, k+1\}))_1,\\ \theta_m(f)(3k+2) &:= f((\{k\}, \{k+1\}))_2.\end{aligned}$$

All sums on the right hand side are to be understood modulo m. Elements of $\operatorname{F\,Hom}(K_2, G)$ are written as pairs of subsets of $V(G)$.

5.3. **Lemma.** *The map θ_m is a well-defined $\mathbb{Z}_2$-map.*

Proof. Since $f((\{k\}, \{k-1\})) \subset f((\{k\}, \{k-1, k+1\}))$, every element of $\theta_m(f)(3k+1)$ is a neighbour of every element of $\theta_m(f)(3k)$, and similarly of every element of $\theta_m(f)(3k+2)$. Since $\theta_m(f)(3k+3) = \theta_m(f)(3(k+1)) = f((\{k+1\}, \{k\}))_2 = f((\{k\}, \{k+1\}))_1$, every element of $\theta_m(f)(3k+2)$ is a neighbour of every element of $\theta_m(f)(3k+3)$, and this calculation also covers the case of the vertices $3m-1$ and 0. Therefore $\theta_m(f)$ is actually a multi-homomorphism from C_{3m} to G. To show that θ_m is equivariant, we calculate

$$\begin{aligned}\theta_m(\tau\cdot f)(3k) &= (\tau\cdot f)((\{k\}, \{k-1\}))_2 = f((\{m-1-k\}, \{m-k\}))_2\\ &= \theta_m(f)(3(m-k-1)+2) = \theta_m(f)(3m-1-3k)\\ &= (\tau\cdot\theta_m(f))(3k),\\ \theta_m(\tau\cdot f)(3k+1) &= (\tau\cdot f)((\{k\}, \{k-1, k+1\}))_1 = f((\{m-1-k\}, \{m-k, m-k-2\}))_1\\ &= \theta_m(f)(3(m-k-1)+1) = \theta_m(f)(3m-1-(3k+1))\\ &= (\tau\cdot\theta_m(f))(3k+1),\\ \theta_m(\tau\cdot f)(3k+2) &= (\tau\cdot f)((\{k\}, \{k+1\}))_2 = f((\{m-1-k\}, \{m-2-k\}))_2\\ &= \theta_m(f)(3(m-k-1)) = \theta_m(f)(3m-1-(3k+2))\\ &= (\tau\cdot\theta_m(f))(3k+2),\end{aligned}$$

which completes the proof. □

5.4. **Proposition.** *Let G be a graph, X a compact triangulable free $\mathbb{Z}_2$-space and*

$$f\colon \mathbb{S}^1_b \times_{\mathbb{Z}_2} X \to_{\mathbb{Z}_2} |\operatorname{Hom}(K_2, G)|$$

an equivariant map. Then there exists an $r \geq 1$ and an equivariant map

$$g\colon X \to_{\mathbb{Z}_2} |\operatorname{Hom}(C_{2r+1}, G)|.$$

5.5. **Remark.** Using notation to be introduced later, this proposition can be seen to provide a poor man's version of a continuous map $\operatorname{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, \operatorname{Hom}(K_2, G)) \to_{\mathbb{Z}_2} \operatorname{colim}_r \operatorname{Hom}(C_{2r+1}, G)$.

Proof of Proposition 5.4. The complexes $\operatorname{Hom}(K_2, C_{2r+1})$ are triangulations of $\mathbb{S}^1_b$. If we take r large enough and K a $\mathbb{Z}_2$-invariant triangulation of X that is fine enough, then there exists a monotone map

$$\mathrm{F}\operatorname{Hom}(K_2, C_{2r+1}) \times \mathrm{F}K \cong \mathrm{F}(\operatorname{Hom}(K_2, C_{2r+1}) \times K) \to_{\mathbb{Z}_2} \mathrm{F}\operatorname{Hom}(K_2, G)$$

which induces an approximation of f. This is adjoint to a montone map

$$\mathrm{F}K \to_{\mathbb{Z}_2} \operatorname{Mon}_{\mathbb{Z}_2}(\mathrm{F}\operatorname{Hom}(K_2, C_{2r+1}), \mathrm{F}\operatorname{Hom}(K_2, G)).$$

Its composition with θ_{2r+1} induces the desired map g. □

5.6. **Corollary.** *Let G be a graph. Then*

$$\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G) \leq 1 + \lim_{r\to\infty} \operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G).$$

Proof. Let $\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G) \geq k+1$. Since $\mathbb{S}^1_b \times_{\mathbb{Z}_2} \mathbb{S}^k$ is a free $(k+1)$-dimensional $\mathbb{Z}_2$-space, there exists a map $\mathbb{S}^1_b \times_{\mathbb{Z}_2} \mathbb{S}^k \to \operatorname{Hom}(K_2, G)$. By the preceding proposition, there exist an r and a map $\mathbb{S}^k \to_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G)$. □

We now set up a framework that will allow us to pass to the limit of the spaces $\operatorname{Hom}(C_m, G)$.

5.7. **Definition.** We define monotone maps

$$\begin{aligned} i_m \colon \mathrm{F}\operatorname{Hom}(C_m, G) &\to \mathrm{F}\operatorname{Hom}(C_{3m}, G), \\ i_m(\phi)(3k) &:= \phi(k-1), \\ i_m(\phi)(3k+1) &:= \phi(k), \\ i_m(\phi)(3k+2) &:= \phi(k+1) \end{aligned}$$

and

$$\begin{aligned} j_m \colon \operatorname{Mon}_{\mathbb{Z}_2}(\mathrm{F}\operatorname{Hom}(K_2, C_m), \mathrm{F}\operatorname{Hom}(K_2, G)) &\to \operatorname{Mon}_{\mathbb{Z}_2}(\mathrm{F}\operatorname{Hom}(K_2, C_{3m}), \mathrm{F}\operatorname{Hom}(K_2, G)), \\ j_m(f)((A, \{3k+1\} \cup B)) &:= f((\{k-1, k+1\}, k)), \\ j_m(f)((A \cup \{3k+1\}, B)) &:= f((k, \{k-1, k+1\})), \\ j_m(f)((\{3k\}, \{3k-1\})) &:= f((\{k-1\}, \{k\})), \\ j_m(f)((\{3k-1\}, \{3k\})) &:= f((\{k\}, \{k-1\})). \end{aligned}$$

5.8. **Proposition.** *$\theta_m \circ \eta_m = i_m$ and $\eta_{3m} \circ \theta_m \leq j_m$.*

Proof. The calculations

$$\begin{aligned} \theta_m(\eta_m(\phi))(3k) &= \eta_m(\phi)((\{k\}, \{k-1\}))_2 = (\phi(k), \phi(k-1))_2 = \phi(k-1) = i_m(\phi)(3k), \\ \theta_m(\eta_m(\phi))(3k+1) &= \eta_m(\phi)((\{k\}, \{k-1, k+1\}))_1 \\ &= (\phi(k), \phi(k-1) \cup \phi(k+1))_1 = \phi(k) = i_m(\phi)(3k+1), \\ \theta_m(\eta_m(\phi))(3k+2) &= \eta_m(\phi)((\{k\}, \{k+1\}))_2 = (\phi(k), \phi(k+1))_2 = \phi(k+1) = i_m(\phi)(3k+2) \end{aligned}$$

prove $\theta_m \circ \eta_m = i_m$. Furthermore

$$\begin{aligned}\eta_{3m}(\theta_m(f))((\{3k\},\{3k-1\})) &= (\theta_m(f)(3k), \theta_m(3(k-1)+2))\\ &= (f((\{k\},\{k-1\}))_2, f((\{k-1\},\{k\}))_2)\\ &= (f((\{k-1\},\{k\}))_1, f((\{k-1\},\{k\}))_2)\\ &= f((\{k-1\},\{k\})) = j_m(f)(\{3k\},\{3k-1\}).\end{aligned}$$

Since $\theta_m(f)(3k-1) \subset \theta_m(f)(3k+1)$ and $\theta_m(f)(3k+3) \subset \theta_m(f)(3k+1)$, we have

$$\begin{aligned}\eta_{3m}(\theta_m(f))((A,\{3k+1\}\cup B)) &\le (\theta_m(f)(3k)\cup\theta_m(f)(3k+2), \theta_m(f)(3k+1))\\ &= (f((\{k\},\{k-1\}))_2 \cup f((\{k\},\{k+1\}))_2, f((\{k\},\{k-1,k+1\}))_1)\\ &\le f((\{k-1,k+1\},\{k\})) = j_m(f)((A,\{3k+1\}\cup B)).\end{aligned}$$

This shows $\eta_{3m}\circ\theta_m \le j_m$. □

5.9. **Definition.** For $m \ge 3$ we define a graph homomorphism

$$\begin{aligned}\kappa_m\colon C_{m+2} &\to C_m\\ 0 &\mapsto m-1,\\ i &\mapsto i-1, \qquad 1\le i\le m,\\ m+1 &\mapsto 0.\end{aligned}$$

Using arithmetic modulo m on the right hand side, this can simply be written as $\kappa_m(i) = i-1$, which makes it clear that this homomorphism commutes with the involutions on C_{m+2} and C_m. For a graph G, we use the induced continuous maps $\operatorname{Hom}(\kappa_m, G)$ to define the colimits (direct limits)

$$\operatorname*{colim}_{m \text{ odd}} \operatorname{Hom}(C_m, G) \qquad \text{and} \qquad \operatorname*{colim}_{m \text{ even}} \operatorname{Hom}(C_m, G).$$

These carry induced $\mathbb{Z}_2$-actions.

The choice of the graph homomorphisms κ_m is not of great importance, as the following lemma shows.

5.10. **Lemma.** *Let G be a graph. The colimit of the diagram of all*

$$|\operatorname{Hom}(C_{3^{n+1}}, G)| \xrightarrow{|\Delta i_{3^{n+1}}|} |\operatorname{Hom}(C_{3^{n+2}}, G)|$$

is $\mathbb{Z}_2$-homotopy equivalent to $\operatorname{colim}_{m\ odd}\operatorname{Hom}(C_m,G)$*, and the colimit of the diagram of all*

$$|\operatorname{Hom}(C_{4\cdot 3^n}, G)| \xrightarrow{|\Delta i_{4\cdot 3^n}|} |\operatorname{Hom}(C_{4\cdot 3^{n+1}}, G)|$$

is $\mathbb{Z}_2$-homotopy equivalent to $\operatorname{colim}_{m\ even}\operatorname{Hom}(C_m,G)$.

Proof. The map i_m is induced by a graph homomorphism $\iota_m\colon C_{3m}\to C_m$. We first consider the case of odd m. It is easy to check that there is a path in $\operatorname{Hom}_{\mathbb{Z}_2}(C_{3(2r+1)}, C_{2r+1})$ connecting ι_{2r+1} and $\kappa_{6r+1}\cdots\kappa_{2r+3}\kappa_{2r+1}$. This induces a $\mathbb{Z}_2$-homotopy between Δi_{2r+1} and $\operatorname{Hom}(\kappa_{6r+1},G)\cdots\operatorname{Hom}(\kappa_{2r+1},G)$ and hence between the homotopy colimits of the corresponding diagrams. Since these diagrams consist of simplicial inclusion maps, hence cofibrations, the natural maps from their homotopy colimits to their colimits are also homotopy equivalences.

For even r we proceed similarly, using a path from $\iota_{6r}\iota_{2r}$ to $\kappa_{18r-2}\cdots\kappa_{2r+2}\kappa_{2r}$. These graph homomorphisms are used because they agree on the vertices 0, $9r-1$, $9r$ and $18r-1$. □

We are now ready to prove the main theorem.

5.11. **Theorem.** *Let G be a graph. Then*

$$\operatorname*{colim}_{m\ odd} \operatorname{Hom}(C_m, G) \simeq_{\mathbb{Z}_2} \operatorname{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, \operatorname{Hom}(K_2, G)),$$

where $\mathbb{Z}_2$ acts on the right hand side by the right action on $\mathbb{S}^1_b$, and

$$\operatorname*{colim}_{m\ even} \operatorname{Hom}(C_m, G) \simeq_{\mathbb{Z}_2} \operatorname{Map}(\mathbb{S}^1_b, \operatorname{Hom}(K_2, G)),$$

where $\mathbb{Z}_2$ acts on the right hand side by the right action on $\mathbb{S}^1_b$ and the action on $\operatorname{Hom}(K_2, G)$.

Proof. We use the diagrams from Lemma 5.10. By Proposition 5.8 the diagram

$$\begin{tikzcd}
{} & {} \\
{|\operatorname{Hom}(C_{3m}, G)|} \arrow[u] \arrow[r, "|\Delta\eta_{3m}|"] & {|\Delta \operatorname{Mon}_{\mathbb{Z}_2}(\operatorname{F}\operatorname{Hom}(K_2, C_{3m}), \operatorname{F}\operatorname{Hom}(K_2, G))|} \arrow[u] \\
{|\operatorname{Hom}(C_m, G)|} \arrow[u, "|\Delta i_m|"] \arrow[r, "|\Delta\eta_m|"] & {|\Delta \operatorname{Mon}_{\mathbb{Z}_2}(\operatorname{F}\operatorname{Hom}(K_2, C_m), \operatorname{F}\operatorname{Hom}(K_2, G))|} \arrow[u, "|\Delta j_m|"'] \arrow[ul, "|\Delta\theta_m|"'] \\
{} \arrow[u] & {} \arrow[u]
\end{tikzcd}$$

commutes up to homotopy and therefore induces a homotopy equivalence between the homotopy colimits of the columns. Since both columns consist of simplicial inclusion maps, which are cofibrations, the homotopy colimits are homotopy equivalent to the colimits.

The 1-dimensional cell-complex $\operatorname{Hom}(K_2, C_{3m})$ can be obtained from $\operatorname{Hom}(K_2, C_m)$ by dividing each 1-cell into three 1-cells. There is a corresponding homeomorphism $|\Delta(\operatorname{F}\operatorname{Hom}(K_2, C_{3m}))| \xrightarrow{\approx} |\Delta(\operatorname{F}\operatorname{Hom}(K_2, C_m))|$ and the map $|\Delta j_m|$ is induced by a map homotopic to it. Thus there is a natural map from the homotopy colimit of the right column to the space $\operatorname{Map}_{\mathbb{Z}_2}(\operatorname{Hom}(K_2, C_3), \operatorname{Hom}(K_2, G))$ in the case of odd m respectively $\operatorname{Map}_{\mathbb{Z}_2}(\operatorname{Hom}(K_2, C_4), \operatorname{Hom}(K_2, G))$ in the case of even m. Using the technique of the proof of Proposition 5.4 we see that these maps are weak homotopy equivalences and hence homotopy equivalences.

All these constructions can be carried out in such a way that the homotopy equivalences are $\mathbb{Z}_2$-maps between free $\mathbb{Z}_2$-spaces and hence $\mathbb{Z}_2$-homotopy equivalences. Finally, $\operatorname{Hom}(K_2, C_3) \approx \mathbb{S}^1_b$ and the quotient of $\operatorname{Hom}(K_2, C_4)$ by the free left $\mathbb{Z}_2$-action is homeomorphic to $\mathbb{S}^1_b$ as a right $\mathbb{Z}_2$-space. □

5.12. **Remark.** In particular, $\operatorname{colim}_r \operatorname{Hom}(C_{2r}, K_{n+2})$ is homotopy equivalent to the free loop space of the n-sphere, a well-studied space. In [Koz05] the cohomology groups of $\operatorname{Hom}(C_m, K_{n+2})$ are determined. Together with an analysis of the maps induced in cohomology by i_m or κ_m this yields an elementary calculation of the cohomology groups of free loop spaces of spheres.

5.13. **Remark.** Since $\operatorname{Hom}(K_2, C_m)$ with the $\mathbb{Z}_2$-action induced by the action on $\mathbb{K}_2$ is homeomorphic to $\mathbb{S}^1$ with the antipodal action for odd m and homeomorphic to $\mathbb{S}^0 \times \mathbb{S}^1$

with an action exchanging the components for even m, we obtain

$$\operatorname*{colim}_r |\mathrm{Hom}(C_{2r}, C_m)| \simeq \mathrm{Map}(\mathbb{S}^1, |\mathrm{Hom}(K_2, C_m)|) \simeq \coprod_{\mathbb{Z}} \mathbb{S}^1,$$

$$\operatorname*{colim}_r |\mathrm{Hom}(C_{2r+1}, C_m)| \simeq \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, |\mathrm{Hom}(K_2, C_m)|) \simeq \begin{cases} \coprod_{\mathbb{Z}} \mathbb{S}^1, & m \text{ odd}, \\ \emptyset, & m \text{ even}. \end{cases}$$

The homotopy types of the spaces $\mathrm{Hom}(C_s, C_m)$ have been determined in [ČK].

5.14. **Remark.** We have seen that $\mathrm{colim}_r \mathrm{Hom}(C_{2r+1}, K_{n+2}) \simeq \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^n)$. There is a canonical map

$$V_{2,n+1} := \{(x,y) \in \mathbb{S}^n \colon \langle x, y \rangle = 0\} \longrightarrow \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^n)$$

which maps (x, y) to a loop following the great circle through x and y at constant speed, starting at x in the direction of y. Among the spaces $\mathrm{Hom}(C_{2r+1}, K_{n+2})$, the space $\mathrm{Hom}(C_5, K_{n+2})$ is special, because it is a manifold [CL05]. It has been conjectured by Csorba [Cso05] and proven in [Sch05b] that there are homeomorphisms

$$\mathrm{Hom}(C_5, K_n) \approx V_{2,n+1}.$$

It should not be difficult to check that these maps can be arranged in a diagram

$$\begin{tikzcd} \mathrm{Hom}(C_5, K_{n+2}) \arrow[r] \arrow[d, "\approx"] & \mathrm{colim}_r \mathrm{Hom}(C_{2r+1}, K_{n+2}) \arrow[d, "\simeq"] \\ V_{2,n+1} \arrow[r] & \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^n) \end{tikzcd}$$

which commutes up to homotopy.

5.15. **Corollary.** *Let G be a graph. If $\mathrm{Hom}(K_2, G)$ is $(k+1)$-connected, then the spaces $\mathrm{colim}_r \mathrm{Hom}(C_{2r+1}, G)$ and $\mathrm{colim}_r \mathrm{Hom}(C_{2r+1}, G)$ are k-connected.*

Proof. Let X be a connected free $\mathbb{Z}_2$-space with non-degenerate basepoint x_0. $\mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, X)$ is homeomorphic to $\{f \in \mathrm{Map}(I, X) \colon f(1) = \tau f(0)\}$. Evaluating at 0 makes this into the total space of a fibration over X with fibre $\{f \in \mathrm{Map}(I, X) \colon f(0) = x_0, f(1) = \tau x_0\}$. The fibre is a fibre of the path fibration over X and hence homotopy equivalent to the loop space ΩX. If now $\pi_k(X) \cong \pi_{k+1}(X) \cong 0$, then from the part

$$0 \cong \pi_{k+1}(X) \cong \pi_k(\Omega X) \to \pi_k(\mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, X)) \to \pi_k(X) \cong 0$$

of the exact homotopy sequence of the fibration it follows that $\pi_k(\mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, X)) \cong 0$. The space $\mathrm{Map}(\mathbb{S}^1, X)$ is also the total space of a fibration with base X and fibre ΩX, so the same conclusion holds. □

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Institut für Mathematik, MA 6-2, Technische Universität Berlin, D-10623 Berlin, Germany

E-mail address: carsten@codimi.de